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AN INTRODUCTION TO HUMAN DESCRIBING FUNCTION AND REMNANT MEASUREMENT IN SINGLE LOOP TRACKING TASKS

by

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FORWARD

This report serves as an introduction to the subject of human describing function measurement. Particular emphasis has been placed upon development of the spectral analysis relations utilized in the describing function measurement techniques. The work was performed by Dr. Hess as part of a research study sponsored by the Air Force Flight Dynamics Laboratory, Air Force Systems Command.

This technical memorandum has been reviewed and is approved.

ABSTRACT

A review of the spectral analysis techniques used in the measurement of human describing functions is presented. The describing function relations for single loop, compensatory tracking tasks are developed. The effect of sinusoidal inputs and finite run lengths are discussed and a brief discussion on mechanization techniques is included.

TABLE OF CONTENTS

I. INTRODUCTION

This memo is a brief introduction to the subject of human describing function measurement with particular emphasis placed upon mathematical formulation. The presentation is intended to serve as a guide to the uninitiated.

The problem to be treated here is one of modeling the human operator via the describing function technique in a single axis, compensatory, tracking task. For discussion of such tasks, see Ref. 1, pp. 7-9. Figure 1 is a block diagram of the compensatory system. Experiments have shown that in tasks such as this, the human operator is nonlinear and time varying in behavior. However, a good deal of success has been achieved in treating him in a quasi-linear fashion[1]. This quasi-linear model assumes that, for the most part, the human behaves in a linear, time invariant manner. This means that the major portion of his response can be attributed to a linear, time invariant operation on his visual stimulus.

Figure 2 is a more detailed block diagram in which the operator has been modeled via a linear describing function and a remnant[1, 2, 3]. The term describing function is preferred to transfer function to emphasize the fact that the model is approximating a nonlinear element and is valid only for particular inputs. Nonetheless, the term transfer function does appear in the literature; e.g., Ref. 4.

Briefly then, the describing function accounts for the operator's linear behavior, the remnant, his nonlinear time varying behavior. Being more specific, in Figure 2:

p(t) represents that portion of the total operator output c(t) which can be obtained by a linear operation on the

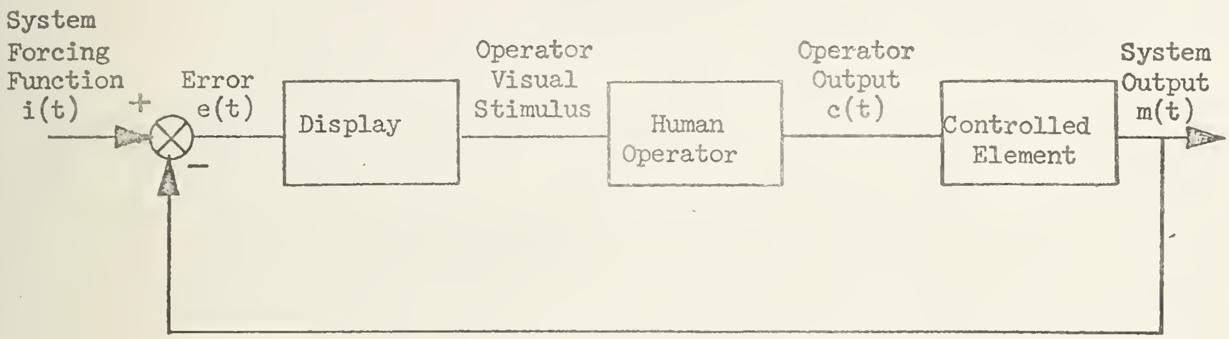

FIGURE 1

The Human Operator in a Compensatory Tracking Control Task

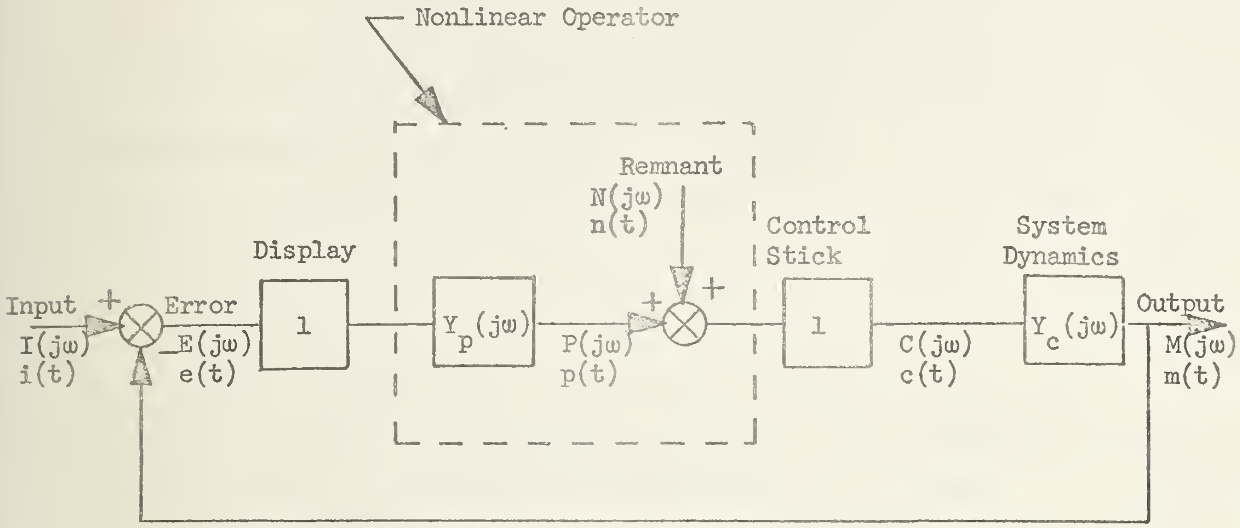

FIGURE 2

Equivalent Block Diagram of Human Operator

in Compensatory Tracking Control Task

visual stimulus e(t).

n(t), the remnant, represents that portion of the total output c(t), which is not linearly correlated with the input i(t).

It should be evident that the utility of this quasi-linear technique depends upon the extent to which the operator is, indeed, linear. If the remnant is relatively large, the describing function technique is of little value in itself.

The modeling problem to be treated here is an empirical one; i.e., to determine $Y_p(j\omega)$ and $\Phi_{nn}(\omega)$ based upon the physical measurements of finite tracking runs in laboratory experiments. The conditions under which these measurements are to be taken are as follows:

1) The conditions implicit in Figure 2 are in effect; i.e., the task is single axis, compensatory, and the system dynamics are linear and time invariant.
2) The input i(t) is truly random or at least random appearing.
3) The operator is well trained. This simply means that his adaptation and learning periods have passed.

Of these three conditions, perhaps the second deserves a brief comment. The reason that the input form must be established is that the operator's dynamics are a definite function of the input; e.g., the operator's behavior when tracking a single sinusoid has been found to be considerably different than when tracking a random or random appearing input. Indeed, the very nature of the task can change when the input is predictable.[5] The rationale behind choosing a random or random appearing input is that it more truly represents the environment

in which human operator models are to be applied; e.g., pilot pitch attitude tracking in the presence of atmospheric turbulence.

II. SPECTRAL ANALYSIS

The techniques used in describing function measurement depend to a great extent upon spectral or harmonic analysis. The brief review which follows is intended to summarize these analytic tools. For a more thorough treatment the reader is referred to Ref. 6, Chapters 2 and 13.

A. Periodic Signals

1. Fourier Series

A periodic signal x(t), with fundamental frequency ω_1 which satisfies the Dirichlet conditions (Ref. 7, p. 248) can be represented by a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X(n)e^{jn\omega_1 t} \qquad (1)$$

$$X(n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_1 t}\,dt \qquad (2)$$

$$T = \frac{2\pi}{\omega_1}$$

2. Autocorrelation and power spectal density

The autocorrelation function for the periodic signal x(t) is defined as

$$\varphi_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \qquad (3)$$

It can be shown that

$$\varphi_{xx}(\tau) = \sum_{n=-\infty}^{\infty} |X(n)|^2 e^{jn\omega_1\tau} \tag{4}$$

Now,

$$\Phi_{xx}(n) = |X(n)|^2 \tag{5}$$

Where $\Phi_{xx}(n)$ is referred to as the power spectral density of the signal $x(t)$. It can be shown that

$$\Phi_{xx}(n) = \frac{1}{T} \int_{-T/2}^{T/2} \varphi_{xx}(\tau) e^{-jn\omega_1\tau}\, d\tau \tag{6}$$

and from eqn. (4)

$$\varphi_{xx}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{xx}(n) e^{jn\omega_1\tau} \tag{7}$$

3. Crosscorrelation and cross power spectral density

The crosscorrelation function for two periodic signals $x(t)$ and $y(t)$ with identical fundamental frequencies ω_1 and which satisfy the Dirichlet conditions is defined as

$$\varphi_{xy}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau)dt \tag{8}$$

It can be shown that

$$\varphi_{xy}(\tau) = \sum_{n=-\infty}^{\infty} \bar{X}(n)Y(n)e^{jn\omega_1\tau} \tag{9}$$

where $\bar{X}(n)$ denotes the complex conjugate of $X(n)$. Now,

$$\Phi_{xy}(n) = \bar{X}(n)Y(n) \tag{10}$$

Where $\Phi_{xy}(n)$ is referred to as the cross power spectral density of the signals $x(t)$ and $y(t)$. One can show

$$\Phi_{xy}(n) = \frac{1}{T}\int_{-T/2}^{T/2} \varphi_{xy}(\tau)e^{-jn\omega_1\tau}\, d\tau \tag{11}$$

and from eqn. (9)

$$\varphi_{xy}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{xy}(n)e^{jn\omega_1\tau} \tag{12}$$

Two periodic signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

B. Transient Signals

1. Fourier Integral

A signal $x(t)$ is said to be transient if

$$\lim_{t\to\infty} x(t) = 0$$

If such a signal

1) satisfies the Dirichlet conditions in any finite interval

2) satisfies the inequality

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

Then the signal can be expressed as a Fourier integral (Ref. 7, p. 279)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \tag{13}$$

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \tag{14}$$

2. Autocorrelation and energy spectral density

The autocorrelation function for the transient signal $x(t)$ is defined as

$$\varphi_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{15}$$

It can be shown that

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 e^{j\omega\tau}\, d\omega \tag{16}$$

Now,

$$\Phi_{xx}(\omega) = |X(j\omega)|^2 \tag{17}$$

Where $\Phi_{xx}(\omega)$ is referred to as the energy spectral density of the signal $x(t)$. It can be shown that

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-j\omega\tau}\, d\tau \tag{18}$$

and from eqn. (16)

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{j\omega\tau}\, d\omega \tag{19}$$

3. Crosscorrelation and cross energy density spectra

The crosscorrelation function for two transient signals $x(t)$ and $y(t)$ each of which satisfies the Dirichlet conditions in every finite interval and which satisfy

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

$$\int_{-\infty}^{\infty} |y(t)|\, dt < \infty$$

is defined as:

$$\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt \qquad (20)$$

It can be shown that

$$\varphi_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X}(j\omega)Y(j\omega)e^{j\omega\tau}\, d\omega \qquad (21)$$

Now

$$\Phi_{xy}(\omega) = \overline{X}(j\omega)Y(j\omega) \qquad (22)$$

Where $\Phi_{xy}(\omega)$ is referred to as the cross energy spectral density of the signals $x(t)$ and $y(t)$.

It can be shown that

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} \varphi_{xy}(\tau)e^{-j\omega\tau}\, d\tau \qquad (23)$$

and from eqn. (21)

$$\varphi_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xy}(\omega)e^{j\omega\tau}\, d\omega \tag{24}$$

Two transient signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

C. Random Signals

1. Fourier Integral

Consider a random signal $x(t)$ as a sample function from a stationary, ergodic random process. Since, in general,

$$\int_{-\infty}^{\infty} |x(t)|\, dt$$

is not finite, one cannot write a Fourier integral representation for $x(t)$.

2. Autocorrelation and Power Spectral Density

The autocorrelation function for the random signal $x(t)$ is defined

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt \tag{25}$$

Now $\varphi_{xx}(\tau)$ can be represented by a Fourier integral since it satisfies the two conditions of section II. B. 1. Hence, it can be shown

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{j\omega\tau}\, d\omega \tag{26}$$

where $\Phi_{xx}(\omega)$ is referred to as the power spectral density of the signal x(t). Here

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-j\omega\tau}\, d\tau \tag{27}$$

3. Crosscorrelation and Cross Power Spectral Density

The crosscorrelation function of two signals x(t) and y(t) which are sample functions from two different random processes, each of which are stationary and ergodic and jointly ergodic, is defined

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)dt \tag{28}$$

Now it can be shown that

$$\varphi_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xy}(\omega)e^{j\omega\tau}\, d\omega \tag{29}$$

where $\Phi_{xy}(\omega)$ is referred to as the cross power spectral density of the two random signals x(t) and y(t). Here

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} \varphi_{xy}(\tau)e^{-j\omega\tau}\, d\tau \tag{30}$$

Two random signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

D. Ergodicity

Briefly, ergodicity is the property of a random process which ensures the equivalence of statistical and time averages. For example, if $x(t)$ is a sample function from an ergodic random process

$$E\left[x(t_1)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

i.e., the expected value of the signal $x(t)$ at time t_1 (a statistical average) equals the time average on the right hand side of the equation.

Ergodicity also ensures that the statistical or spectral measures (they're now equivalent) taken from a single experiment are indeed representative of the random process under study.

E. Relations for Linear Systems

Ref. 6, p. 333 shows that, given a linear, time-invariant system, with weighting function (impulse response) $h(t)$, input $x(t)$, and output $y(t)$, where $x(t)$ and $y(t)$ are the random functions alluded to in section II. C. 2, 3

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\Phi_{yy}(\omega) = \Phi_{xx}(\omega) \left|H(j\omega)\right|^2 \qquad (31)$$

Also

$$H(j\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)} \tag{32}$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt = \int_{0}^{\infty} h(t)e^{-j\omega t}\, dt = H(s)\Big|_{s=j\omega} \tag{33}$$

and H(s) is the system transfer function.

III. DESCRIBING FUNCTION RELATIONS

A. Problem Statement

Consider Figure 2 again. Let the input i(t) be a sample function from an ergodic random process. On the basis of measurements of finite time histories of the signals i(t), e(t), c(t) and m(t), one wishes to find

1) The operator's linear describing function $Y_p(j\omega)$.

2) The power spectral density of the remnant signal n(t), $\Phi_{nn}(\omega)$.

B. Finite Run Length

Obviously, for the purposes of practical data collection, only finite time histories of the system signals of Figure 2 are available. But referring to section II. C., one sees that integration over infinite time is implied in the various spectral measures of the random signals. The question of just how long the histories must be in order to ensure accurate spectral measurements naturally arises. Here it will simply be stated that T, the run length, must be long enough to ensure accurate determination of the autocorrelation functions.

Consider the system input, i(t), in Figure 2. Define

$$i_T(t) = i(t) \quad -T < t < T$$

$$i_T(t) = 0 \qquad \text{other } t$$

Now let

$$\varphi_{ii}(\tau)_T = \frac{1}{2T}\int_{-T}^{T} i_T(t)\,i_T(t+\tau)\,dt \qquad (34)$$

Note that

$$\lim_{T\to\infty} \varphi_{ii}(\tau)_T = \varphi_{ii}(\tau) \qquad (35)$$

Also define

$$\Phi_{ii}(\omega)_T = \int_{-T}^{T} \varphi_{ii}(\tau)_T e^{-j\omega\tau}\, d\tau \qquad (36)$$

Note that

$$\lim_{T\to\infty} \Phi_{ii}(\omega)_T = \Phi_{ii}(\omega) \qquad (37)$$

In light of the definition of $i_T(t)$, eqns. (34) and (36) can be written

$$\varphi_{ii}(\tau)_T = \frac{1}{2T}\int_{-\infty}^{\infty} i_T(t)i_T(t+\tau)dt \qquad (38)$$

$$\Phi_{ii}(\omega)_T = \int_{-\infty}^{\infty} \varphi_{ii}(\tau)_T e^{-j\omega\tau}\, d\tau \qquad (39)$$

Finally, since $i_T(t)$ looks very much like a transient signal, one can assume that the conditions of section II. B. 1 are applicable and that the Fourier integral (or transform) of $i_T(t)$ can be written; i.e.,

$$I(j\omega) = \int_{-\infty}^{\infty} i_T(t)e^{-j\omega t}\, dt \qquad (40)$$

Furthermore, from eqns. (38) and (39)

$$\Phi_{ii}(\omega)_T = \int_{-\infty}^{\infty} \frac{1}{2T} \int_{-\infty}^{\infty} i_T(t) i_T(t+\tau) e^{-j\omega\tau}\, dt d\tau \tag{41}$$

Now defining a new variable u as

$$u = t + \tau$$

one can write

$$\Phi_{ii}(\omega)_T = \frac{1}{2T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i_T(t) i_T(u) e^{-j\omega(u-t)}\, dt du \tag{42}$$

$$= \frac{1}{2T}\left[\int_{-\infty}^{\infty} i_T(t) e^{j\omega t}\, dt \right]\left[\int_{-\infty}^{\infty} i_T(u) e^{-j\omega u}\, du \right]$$

$$= \frac{1}{2T}\left[\bar{I}(j\omega) \right]\left[I(j\omega) \right] = \frac{1}{2T} \left| I(j\omega) \right|^2 \tag{43}$$

from eqns. (37) and (43)

$$\Phi_{ii}(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T} \left| I(j\omega) \right|^2 \right] \tag{44}$$

Likewise, one can show

$$\Phi_{ic}(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T} \bar{I}(j\omega) C(j\omega) \right] \tag{45}$$

The limiting processes of eqns. (44) and (45) should be interpreted as allowing T to be large enough to ensure accurate spectral measurements but finite so that eqn. (40) is still valid.

C. Spectral Relations

Referring again to the block diagram of Figure 2

$$E(j\omega) = I(j\omega) - M(j\omega)$$

$$= I(j\omega) - \left[N(j\omega) + E(j\omega)Y_p(j\omega) \right] Y_c(j\omega)$$

or

$$E(j\omega) = \frac{I(j\omega) - N(j\omega)Y_c(j\omega)}{1 + Y_c(j\omega)Y_p(j\omega)}$$

Now

$$\overline{I}(j\omega)E(j\omega) = \frac{\overline{I}(j\omega)\left[I(j\omega) - N(j\omega)Y_c(j\omega) \right]}{1 + Y_c(j\omega)Y_p(j\omega)} \tag{46}$$

In like manner, dropping the $(j\omega)$ notation, one can show

$$\overline{I}C = \frac{\overline{I}\left[Y_pI + Y_cN \right]}{1 + Y_cY_p} \tag{47}$$

$$\overline{E}C = \frac{\left[\overline{I} - \overline{Y}_c\overline{N} \right]\left[Y_cI + N \right]}{\left[1 + \overline{Y}_c\overline{Y}_p \right]\left[1 + Y_cY_p \right]} \tag{48}$$

$$\overline{E}E = \frac{\left[\overline{I} - \overline{Y}_c\overline{N} \right]\left[I - Y_cN \right]}{\left[1 + \overline{Y}_c\overline{Y}_p \right]\left[1 + Y_cY_p \right]} \tag{49}$$

$$\overline{C}C = \frac{\left[\overline{N} + \overline{Y}_p\overline{I}\right]\left[N + Y_p I\right]}{\left[1 + \overline{Y}_c\overline{Y}_p\right]\left[1 + Y_c Y_p\right]} \tag{50}$$

Now as in eqns. (44) and (45)

$$\Phi_{ie} = \lim_{T\to\infty} \frac{1}{2T}\left[\overline{I}E\right] = \lim_{T\to\infty}\frac{1}{2T} \frac{\overline{I}\left[I - Y_c N\right]}{1 + Y_c Y_p}$$

$$= \frac{\lim_{T\to\infty} \frac{1}{2T}\overline{I}I - Y_c \lim_{T\to\infty}\frac{1}{2T}\overline{I}N}{1 + Y_c Y_p}$$

$$= \frac{\Phi_{ii} - \Phi_{in}Y_c}{1 + Y_c Y_p} \tag{51}$$

Now

$$\Phi_{in} = \int_{-\infty}^{\infty} \varphi_{in}(\tau)e^{-j\omega\tau}\, d\tau \tag{52}$$

But since the remnant, n(t), is, by definition, linearly uncorrelated with the input, i(t),

$$\varphi_{in}(\tau) = 0 \quad \text{for all } \tau$$

Thus

$$\Phi_{in}(\omega) = 0$$

Finally

$$\Phi_{ie}(\omega) = \frac{\Phi_{ii}(\omega)}{1 + Y_c Y_p(j\omega)} \tag{53}$$

Likewise

$$\Phi_{ic}(\omega) = \frac{Y_p(j\omega)\Phi_{ii}(\omega)}{1 + Y_c Y_p(j\omega)} \tag{54}$$

$$\Phi_{ec}(\omega) = \frac{Y_p(j\omega)\Phi_{ii}(\omega) - \overline{Y}_c(j\omega)\Phi_{nn}(\omega)}{\left|1 + Y_c Y_p(j\omega)\right|^2} \tag{55}$$

$$\Phi_{ee}(\omega) = \frac{\Phi_{ii}(\omega) + \left|Y_c(j\omega)\right|^2 \Phi_{nn}(\omega)}{\left|1 + Y_c Y_p(j\omega)\right|^2} \tag{56}$$

$$\Phi_{cc}(\omega) = \frac{\Phi_{nn}(\omega) + \left|Y_p(j\omega)\right|^2 \Phi_{ii}(\omega)}{\left|1 + Y_c Y_p(j\omega)\right|^2} \tag{57}$$

Now from the above

$$Y_p(j\omega) = \frac{\Phi_{ic}(\omega)}{\Phi_{ie}(\omega)} \tag{58}$$

$$\Phi_{nn}(\omega) = \left|1 + Y_c Y_p(j\omega)\right|^2 \Phi_{cc}(\omega) - \left|Y_p(j\omega)\right|^2 \Phi_{ii}(\omega) \tag{59}$$

Equations (58) and (59) form the basis for the describing function measurement techniques to be discussed. Before proceeding, it may be interesting to point out a pitfall in describing function measurement. Occasionally in the literature; e.g., Ref. 8, $Y_p(j\omega)$ is given as

$$Y_p(j\omega) = \frac{\Phi_{ec}(\omega)}{\Phi_{ee}(\omega)} \tag{60}$$

From the preceding results, however, one can see that

$$\frac{\Phi_{ec}(\omega)}{\Phi_{ee}(\omega)} = \frac{Y_p(j\omega)\Phi_{ii}(\omega) - \overline{Y}_c(j\omega)\Phi_{nn}(\omega)}{\Phi_{ii}(\omega) + |Y_c(j\omega)|^2\Phi_{nn}(\omega)} \tag{61}$$

This relation represents $Y_p(j\omega)$ only when the remnant is zero, i.e. $\Phi_{nn}(\omega) = 0$. Thus $Y_p(j\omega)$ measurement via eqn. (60) will, in general, be contaminated by remnant, whereas measurement via eqn. (58) will not.

D. Sinusoidal Inputs

1. Introduction

The describing function relations in section III. C were predicated on the existence of a truly random input i(t). Often in experimental work the random input is replaced by a random appearing input, mechanized as the sum of sine waves; i.e.,

$$i(t) = \sum_{k=1}^{N} A_k \sin\omega_k t$$

Here the ω_k are chosen so as to be non-commensurable (no frequency is an integral multiple of another), and roughly evenly spaced on a logarithmic scale. In addition, the ω_k are selected so that in a finite experimental run length, say 100 seconds, all the constituent sine waves in i(t) will have completed an integral number of cycles. Finally, the ω_k are chosen to lie within the range of interest of human response work; i.e.,

$$0.1 < \omega_k < 20 \text{ rad/sec}$$

As few as four sine waves can be utilized to create a random appearing input; e.g. Ref. 1, p. 78.

2. Comparison of Spectral Relations

The spectral relations for pure sinusoids and finite time histories of pure random signals deserve some comparison. This is done in Table I. The reason that this comparison is important is that signals in tracking tasks with sinusoidal inputs contain both random and periodic components. The random component stems from the remnant (see Ref. 2, p. 127), the periodic component stems from the input. Thus the question naturally arises as to which definitions to use in computing cross power and power spectral densities. The answer is

1) Use column A in computing spectral measures at the input frequencies.

2) Use column B in computing spectral measures at other than input frequencies.

TABLE I

Comparison of Random and Sinusoidal Spectral Relations

A	B
Sinusoids	Finite, Random
$x(t) = \sum_{n=-\infty}^{\infty} X(n) e^{j\omega_1 t}$	$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$
$X(n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_1 t}\, dt$	$X(j\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t}\, dt$
$\varphi_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt$	$\varphi_{xx}(\tau)_T = \frac{1}{2T}\int_{-T}^{T} x_T(t)x_T(t+\tau)dt$
$\Phi_{xx}(n) = \frac{1}{T}\int_{-T/2}^{T/2} \varphi_{xx}(\tau) e^{-jn\omega_1 \tau}\, d\tau$	$\Phi_{xx}(\omega)_T = \int_{-T}^{T} \varphi_{xx}(\tau)_T e^{-j\omega\tau}\, d\tau$
$\Phi_{xx}(n) = \lvert X(n)\rvert^2$	$\Phi_{xx}(\omega)_T = \lim_{T\to\infty}\left[\frac{1}{2T}\lvert X(j\omega)\rvert^2\right]$
$\Phi_{xy}(n) = \overline{X}(n)Y(n)$	$\Phi_{xy}(\omega)_T = \lim_{T\to\infty}\left[\frac{1}{2T}\overline{X}(j\omega)Y(j\omega)\right]$

Here T can be any multiple of the period of x(t)

The reasoning behind statements (1) and (2) is quite simple. At other than input frequencies, all the power in the system signals stems from the random remnant, thus the natural selection of column B. The measurements made at input frequencies will involve cross spectral measures with the periodic input, thus the choice of column A.

Referring to Table I, one can see that if T contains an integral number of periods of the periodic signal the following relations hold between columns A and B:

$$X(j\omega_k) = 2TX(n_k)$$
$$\Phi_{xx}(\omega_k)_T = 2T\Phi_{xx}(n_k) \tag{62}$$

where ω_k denotes frequencies existing in the sinusoidal input, and $X(n_k)$ the Fourier transform of x(t) evaluated at ω_k.

3. Advantages of Sinusoidal Inputs

The chief advantage in utilizing sinusoidal inputs lies in the simplification of the spectral measurements of eqns. (58) and (59). Recalling eqn. (58):

$$Y_p(j\omega) = \frac{\Phi_{ic}(\omega)}{\Phi_{ie}(\omega)}$$

At the input frequencies, and with T large and containing an integral number of periods of i(t)

$$Y_p(j\omega_k) = \frac{\Phi_{ic}(\omega_k)}{\Phi_{ie}(\omega_k)} \doteq \frac{\Phi_{ic}(\omega_k)_T}{\Phi_{ie}(\omega_k)_T} = \frac{2T\Phi_{ic}(n_k)}{2T\Phi_{ie}(n_k)}$$

or

$$Y_p(j\omega_k) \doteq \frac{\Phi_{ic}(n_k)}{\Phi_{ie}(n_k)} = \frac{\bar{I}(n_k)C(n_k)}{\bar{I}(n_k)E(n_k)}$$

or

$$Y_p(j\omega_k) = \frac{C(n_k)}{E(n_k)} \tag{63}$$

Thus, with the sinusoidal input, the necessity of measuring cross power spectral densities is sidestepped. Only the Fourier integrals (or transforms) of the signals c(t) and i(t) are needed and this is a preferable alternative to cross spectral measurements.

A second advantage to sinusoidal input experiments lies in the simplification of eqn. (59). That equation reads

$$\Phi_{nn}(\omega) = \left|1 + Y_c Y_p(j\omega)\right|^2 \Phi_{cc}(\omega) - \left|Y_p(j\omega)\right|^2 \Phi_{ii}(\omega)$$

The function $\Phi_{nn}(\omega)$ is evaluated by taking the difference between two quantities which, in practice, are comparatively large and nearly equal. This leads to inaccuracies in calculating $\Phi_{nn}(\omega)$. If, however, one utilizes sinusoidal inputs and measures $\Phi_{nn}(\omega)$ at other than input frequencies, one can write

$$\Phi_{nn}(\omega_h) = \left|1 + Y_c Y_p(j\omega_h)\right|^2 \Phi_{cc}(\omega_h) \tag{64}$$

since $\Phi_{ii}(\omega_h) = 0$.

where ω_h indicates frequencies other than input frequencies.

Note that at the frequencies ω_h

$$Y_p(j\omega_h) \neq \frac{C(n_k)}{E(n_k)}$$

although some literature seems to indicate otherwise[10]. An estimate of $Y_p(j\omega_h)$ to be used in eqn. (64) can be obtained by interpolating between values of $Y_p(j\omega)$ found at the two input frequencies to either side of ω_h .

4. Disadvantages of Sinusoidal Inputs

Probably the chief disadvantage of sinusoidal input experiments is that measurement of $Y_p(j\omega)$ can be made only at input frequencies. Whereas with inputs with continuous spectra, the cross spectral densities utilized in defining $Y_p(j\omega)$ can be made at virtually any frequency desired.

Another, minor disadvantage of sinusoidal inputs is that some "shaping" of the sum of sinusoids is necessary to simulate the continuous spectra inputs/disturbances encountered in actual man-machine systems. A quite satisfactory way of doing this is utilized in Ref. 11.

IV. MECHANIZATION

The methods of performing the measurements indicated in Eqs. (58) and (59) of Section III-C are quite varied. Perhaps just a word about three of them is in order.

A. Autocorrelation-Spectral Density

Reference 12 exemplifies an approach wherein the autocorrelation functions $\varphi_{ic}(\tau)$ and $\varphi_{ie}(\tau)$ are first calculated and the cross power spectral densities $\Phi_{ic}(\omega)$ and $\Phi_{ie}(\omega)$ obtained from these. Tape recordings of the signals are digitized and the correlation and spectral density calculations are performed digitally.

B. Analog Fourier Transformation

The Systems Technology Inc. Describing Function Analyzer is an analog device which computes the Fourier transform of any system signal[9]. The analyzer also provides the system input as the sum of five sinusoids. The recommended measurement is

$$\frac{E(n_k)}{I(n_k)} = \frac{\Phi_{ie}(n_k)}{\Phi_{ii}(n_k)} = \frac{\frac{1}{2T}\Phi_{ie}(\omega_k)_T}{\frac{1}{2T}\Phi_{ii}(\omega_k)_T} \doteq \frac{\Phi_{ie}(\omega_k)}{\Phi_{ii}(\omega_k)}$$

but from Section III-C

$$\frac{\Phi_{ie}(\omega_k)}{\Phi_{ii}(\omega_k)} = \frac{1}{1 + Y_c Y_p(j\omega_k)}$$

Thus, knowing $Y_c(j\omega_k)$, one can find $Y_p(j\omega_k)$ by measuring $\dfrac{E(n_k)}{I(n_k)}$.

C. Hybrid Fourier Transformation

Reference 10 offers an on-line hybrid (analog-digital) mechanization in which the Fourier transformation of the signals

$c(t)$ and $e(t)$ is performed on a hybrid computer using Fast Fourier transform techniques. Here

$$Y_p(j\omega_k) = \frac{C(n_k)}{E(n_k)}$$

V. COMMENT

The author hopes that this memo has served as a readable introduction to the subject of describing function measurement. The treatment was not intended to be exhaustive but rather to highlight the salient features of measurement techniques based upon spectral analysis. If the reader has obtained a fair grasp of the material, he should be able to move on to the subject of describing function measurements in multiloop tasks. Reference 13 offers a good introduction to the subject.

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